

ESTIMATING AVERAGES OF ORDER STATISTICS OF BIVARIATE FUNCTIONS

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ABSTRACT. We prove uniform estimates for the expected value of averages of order statistics of bivariate functions in terms of their largest values by a direct analysis. As an application, uniform estimates for the expected value of averages of order statistics of sequences of independent random variables in terms of Orlicz norms are obtained. In the case where the bivariate functions are matrices, we provide a “minimal” probability space which allows us to C -embed certain Orlicz spaces ℓ_M^n into $\ell_1^{cn^3}$, $c, C > 0$ being absolute constants.

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1. INTRODUCTION AND MAIN RESULTS

In the series of papers [7, 8, 9, 10, 11], sequences of random variables and their order statistics were studied in several different settings and the obtained results were applied successfully to the local theory of convex bodies. In [7], the authors studied expressions of the form

$$\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X_i|, \quad 1 \leq \ell \leq n, \quad (1.1)$$

with independent identically distributed (iid) random variables X_i , $i = 1, \dots, n$ and real numbers x_i , $i = 1, \dots, n$. Here, $\text{k-max}_{1 \leq i \leq n} X_i(\omega)$ is the k -th order statistic of a statistical sample of size n , which is equal to its k -th largest value. Besides being fundamental tools in statistics with various applications in, e.g., compressed

sensing [4, 17], wireless networks [15], and data streams [32], order statistics of random samples appear naturally in Banach space theory, e.g., in computations of the distribution of eigenvalues of random matrices [19, 27], and in calculating sharp bounds for the expectation of the supremum of Gaussian processes indexed by certain interpolated bodies [6]. For general information on order statistics, we refer the reader to [5].

Especially we would like to point out that the results that were obtained in [7], in particular the estimates for (1.1), were crucial to obtain estimates for various parameters associated to the local theory of convex bodies [8], e.g., type and cotype constants, p -summing norms, volume ratios, and projection constants.

An integral tool in [7, 8] and thus in [9, 10, 11] are combinatorial estimates going back to S. Kwapien and C. Schütt [13, 14]. Those estimates relate an average (over the group of permutations) of the largest order statistic of a matrix a to the average of its largest entries. To be more precise, it was shown that for all $a \in \mathbb{R}^{n \times n}$

$$\frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \max_{1 \leq i \leq n} |a_{i\pi(i)}| \simeq \frac{1}{n} \sum_{k=1}^n s(k),$$

where $s(k)$ is the k -th largest entry of the matrix consisting of the absolute values of a and \mathfrak{S}_n is the symmetric group on $\{1, \dots, n\}$. In [16], this was established in the following setting: under some assumptions on the normalized counting measure \mathbb{P} on a collection G of maps from $I = \{1, \dots, n\}$ to $J = \{1, \dots, N\}$, we have that for every matrix $a \in \mathbb{R}^{n \times N}$ and every $\ell \leq n$,

$$\frac{c}{N} \sum_{j=1}^{\ell N} s(j) \leq \int_G \sum_{k=1}^{\ell} \mathbf{k}\text{-max}_{1 \leq i \leq n} |a_{ig(i)}| d\mathbb{P}(g) \leq \frac{C}{N} \sum_{j=1}^{\ell N} s(j), \quad (1.2)$$

where c and C are positive constants only depending on G . Special choices for G so that (1.2) holds, include the symmetric group \mathfrak{S}_n and $\{1, \dots, n\}^{\{1, \dots, n\}}$. Those estimates were then used to deduce similar combinatorial estimates for ℓ_p norms.

In this work we extend our results from [16] and study averages of order statistics of bivariate functions $a : \{1, \dots, n\} \times \Omega \rightarrow \mathbb{R}$, where $(\Omega, \mathcal{F}, \mu)$ is an arbitrary probability space. In this setting, G will be a collection of maps from $I = \{1, \dots, n\}$ to the probability space $(\Omega, \mathcal{F}, \mu)$. We denote the decreasing rearrangement of a by a^* . Our main result is the following:

Theorem 1.1. *Let $n \in \mathbb{N}$, G be a collection of maps from $I = \{1, \dots, n\}$ to the probability space (Ω, μ) , $C_G \geq 1$ be a constant only depending on G , and \mathbb{P} be a probability measure on G . Assume that for all $i \in I$, all different indices $i_1, i_2 \in I$ and all measurable sets $A, A_1, A_2 \subset \Omega$,*

- (i) $\mathbb{P}(g(i) \in A) = \mu(A)$,
- (ii) $\mathbb{P}(g(i_1) \in A_1, g(i_2) \in A_2) \leq C_G \mu(A_1) \mu(A_2)$.

Then, for every measurable function $a : I \times \Omega \rightarrow \mathbb{R}$ and for every $\ell \leq n$,

$$c \cdot \int_0^\ell a^*(t) dt \leq \int_G \sum_{k=1}^{\ell} \mathbf{k}\text{-max}_{1 \leq i \leq n} |a(i, g(i))| d\mathbb{P}(g) \leq C \cdot \int_0^\ell a^*(t) dt, \quad (1.3)$$

where $1/c = 48(1 + 2C_G)^2$ and $C = 6(1 + 2C_G)$.

As a direct consequence, when the bivariate functions are matrices and \mathbb{P} is the normalized counting measure on G , we obtain one of the main results in [16, Theorem 1.1]:

Corollary 1.2. *Let $n, N \in \mathbb{N}$ and $a \in \mathbb{R}^{n \times N}$. Let G be a collection of maps from $I = \{1, \dots, n\}$ to $J = \{1, \dots, N\}$ and $C_G > 0$ be a constant only depending on G . Assume that for all $i \in I$, $j \in J$ and all different pairs $(i_1, j_1), (i_2, j_2) \in I \times J$*

- (i) $\mathbb{P}(\{g \in G : g(i) = j\}) = 1/N$,
- (ii) $\mathbb{P}(\{g \in G : g(i_1) = j_1, g(i_2) = j_2\}) \leq C_G/N^2$.

Then, for every $\ell \leq n$,

$$\frac{c}{N} \sum_{j=1}^{\ell N} s(j) \leq \int_G \sum_{k=1}^{\ell} \mathbf{k}\text{-max}_{1 \leq i \leq n} |a_{ig(i)}| d\mathbb{P}(g) \leq \frac{C}{N} \sum_{j=1}^{\ell N} s(j), \quad (1.4)$$

where $1/c = 48(1 + 2C_G)^2$ and $C = 6(1 + 2C_G)$.

In this work we also present an example of a set of maps, say G_0 , with a minimal number of elements satisfying conditions (i) and (ii) in Corollary 1.2, thus guaranteeing that the inequalities in (1.4) hold. When $N = n$, the cardinality of G_0 is n^2 . The set of maps provided here is based on finite fields of n elements, where n is a power of a prime number. It is not hard to see that if G and \mathbb{P} satisfy conditions (i) and (ii) of Corollary 1.2 for some constant $C_G \geq 1$, then G consists of at least $\frac{n^2}{C_G}$ elements.

We then apply this result to obtain the following:

Theorem 1.3. *There exist constants $c, C > 0$ such that for all $n \in \mathbb{N}$ and every strictly convex, twice differentiable Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ that is strictly 2-concave and satisfies $M^*(1) = 1$, we have that $\ell_M^n \xrightarrow{C} \ell_1^{cn^3}$.*

We also provide an application of Theorem 1.1 to sequences of iid random variables to obtain estimates for (1.1). Those estimates are in terms of Orlicz norms and recover Corollaries 2 and 3 of [7]. To be more precise, we prove the following:

Theorem 1.4. *Let X_1, \dots, X_n be a sequence of iid random variables with $\mathbb{E}|X_1| < \infty$. Let $1 \leq \ell \leq n$ and M be the N -function given by*

$$M^*\left(\int_0^\beta X^*(z) dz\right) = \frac{\beta}{\ell}, \quad 0 \leq \beta \leq 1. \quad (1.5)$$

Then, for all $x \in \mathbb{R}^n$,

$$c\|x\|_M \leq \mathbb{E} \sum_{k=1}^{\ell} \mathbf{k}\text{-max}_{1 \leq i \leq n} |x_i X_i| \leq C\|x\|_M,$$

where $c, C > 0$ are absolute constants.

Let M^* be given as in (1.5). Then, for all $s \geq 0$,

$$M(s) = \int_0^s \int_{|X| \geq 1/(t\ell)} |X| d\mathbb{P} dt.$$

For $\ell = 1$, this was shown in [11, pp. 4-5]. A simple calculation shows that it holds for general ℓ as well. Therefore, we indeed obtain Corollaries 2 and 3 of [7].

While in [7] the proof involves estimates for the largest order statistic of a matrix and makes use of combinatorial results of [13, 14] in a crucial way, our approach is based on a purely probabilistic and direct analysis of (1.1) (Theorem 1.1), and is interesting in its own right.

The organization of the paper is as follows. Section 2 serves the purpose of introducing notations and preliminary results that we use throughout the paper, where the measure theoretic ones are especially important for the proof of the main theorem. Section 3 contains the proof of Theorem 1.1. This is done by reducing the problem to the case of functions only taking values in $\{0, 1\}$ and showing the result for this subclass of functions. Section 4 contains the application of Theorem 1.1 to sequences of iid random variables and thus the proof of Theorem 1.4. In Section 5, we present the minimal set of maps G_0 needed to guarantee the inequalities in (1.4) and give a proof of the embedding result Theorem 1.3.

2. NOTATION AND PRELIMINARIES

Given a random variable $X : \Omega \rightarrow \mathbb{R}$ on a measure space $(\Omega, \mathcal{A}, \mu)$, we define its distribution function $F_X(t) := \mu(\{\omega \in \Omega : |X(\omega)| > t\})$, $t \geq 0$. The decreasing rearrangement of X is then defined for all $t \geq 0$ by

$$X^*(t) = \inf\{s \geq 0 : F_X(s) \leq t\},$$

where we use the convention that $\inf \emptyset = \infty$. Note that if F_X is continuous and strictly decreasing, then X^* is simply the inverse of F_X . Moreover, notice that X and X^* are equimeasurable, i.e.,

$$\mu(X \in A) = \lambda(X^* \in A)$$

for all measurable subsets $A \subset \mathbb{R}$, where λ denotes the Lebesgue measure. By $\mathbb{1}_A$ we denote the characteristic function of a set A .

A convex function $M : [0, \infty) \rightarrow [0, \infty)$ is called an Orlicz function, if $M(0) = 0$ and if M is not constant. An Orlicz function (as we define it) is bijective and continuous on $[0, \infty)$. Given an Orlicz function M , we define its conjugate function M^* via the Legendre transform,

$$M^*(x) = \sup_{t \in [0, \infty)} (xt - M(t)).$$

We have $M = M^{**}$, since the Legendre transform is an involution. Note that M^* is again an Orlicz function if M is an N -function, i.e., if additionally

$$\lim_{x \rightarrow 0} M(x)/x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} M(x)/x = \infty.$$

Given an Orlicz function M and a measure space (Γ, ψ) , the Orlicz space $L_M(\Gamma)$ is the space of all (equivalence classes of) measurable, real valued functions f on Γ such that

$$\int_{\Gamma} M(|f|/\lambda) d\psi < \infty,$$

for some $\lambda > 0$. We equip $L_M(\Gamma)$ with the Luxemburg norm

$$\|f\|_M = \inf \left\{ \lambda > 0 : \int_{\Gamma} M(|f|/\lambda) d\psi \leq 1 \right\}.$$

The closed unit ball of the space L_M will be denoted by B_M . Note also that if M is an N -function, we have

$$\|f\|_M \leq \sup_{g \in B_{M^*}} \int_{\Gamma} f \cdot g d\psi \leq 2\|f\|_M. \quad (2.1)$$

For a detailed and thorough introduction to Orlicz spaces, cf. eg. [26] or [18].

Another result we will use is Paley-Zygmund's inequality:

Theorem 2.1 (Paley-Zygmund). *For every non-negative random variable Z and every number $0 < \theta < 1$, we have*

$$\mathbb{P}(Z \geq \theta \cdot \mathbb{E}Z) \geq (1 - \theta)^2 \frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2}.$$

Moreover, we will need the following measure theoretic results:

Theorem 2.2 (Sierpiński). *Let (R, \mathcal{R}, ρ) be a non-atomic measure space with $\rho(R) = c$. Then there exists a function $f : [0, c] \rightarrow \mathcal{R}$ satisfying*

- (i) $f(t) \subset f(s)$ for $0 \leq t \leq s \leq c$,
- (ii) $\rho(f(s)) = s$ for $0 \leq s \leq c$.

Sierpiński's theorem allows us to construct to a given measurable function a new one that is constant only on sets of measure zero and which has the same ordering.

Proposition 2.3. *Let (R, \mathcal{R}, ρ) be a finite measure space. For every measurable function $a : R \rightarrow [0, \infty)$ there exists a measurable function $b : R \rightarrow [0, \infty)$ with the following properties:*

- (i) for all $x \in [0, \infty)$ we either have $\rho(b = x) = 0$ or $\{b = x\}$ is an atom.
- (ii) for all $s, t \in R$, we have $a(s) > a(t)$ implies that $b(s) > b(t)$.

Proof. Before we begin with the construction of the function b satisfying properties (i) and (ii), we sketch its idea.

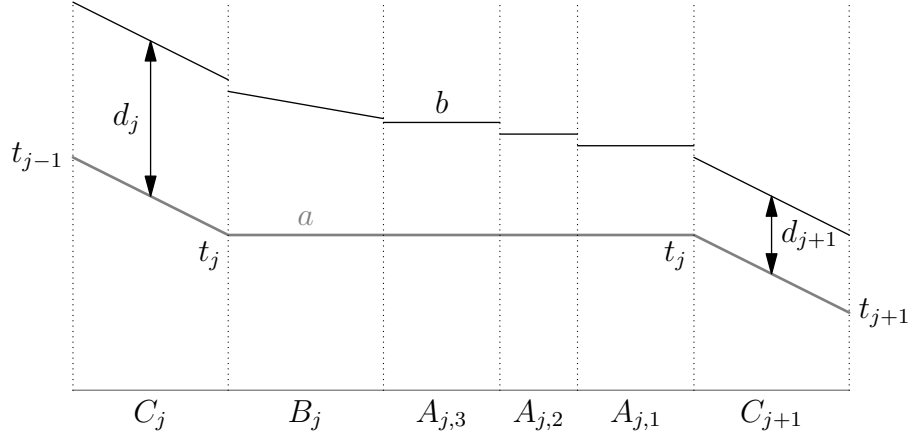


FIGURE 1. Construction of b .

First, we consider the sets $\{a = t_j\}$ for those t_j such that $\rho(a = t_j) > 0$. We decompose each of those sets into atoms $A_{j,k}$ of \mathcal{R} and a continuous part B_j . We define our function b on $\{a = t_j\}$ in such way that it takes different values on each of the atoms $A_{j,k}$. On the continuous part B_j , it is defined in such a way that $\rho(\{b = t\} \cap B_j) = 0$ for all t , where we use Sierpiński's theorem.

Let $(t_j)_{j \in \mathcal{N}}$ be the decreasing sequence of all numbers t such that

$$\rho(a = t) > 0.$$

Note that there are at most countably many t 's with this property, i.e., we can choose $\mathcal{N} = \{1, \dots, N\}$ for some $N \in \mathbb{N}$ or $\mathcal{N} = \mathbb{N}$. Additionally, we set $t_0 = \infty$

and define the sets

$$C_j = \{t_j < a < t_{j-1}\}, \quad j \in \mathcal{N} \quad \text{and} \quad D = \{a < \inf_{j \in \mathcal{N}} t_j\}.$$

First, we specify the function b on the sets C_j , $j \in \mathcal{N}$ and D by

$$b(s) = a(s) + d_j, \quad s \in C_j \quad \text{and} \quad b(s) = a(s), \quad s \in D, \quad (2.2)$$

where $d_j = \sum_{i \in \mathcal{N}, i \geq j} 2^{-i}$. Note that we have

$$\{a = t_j\} = \bigcup_{k \in \mathcal{M}_j} A_{j,k} \cup B_j, \quad j \in \mathcal{N},$$

where $\{A_{j,k}\}$ are atoms, B_j is non-atomic and \mathcal{M}_j is either $\{1, \dots, M_j\}$ or \mathbb{N} . The offset d_j introduced in (2.2) allows us now to define b on $\{a = t_j\}$ such that it takes different values on each of the atoms $A_{j,k}$ and such that $\rho(\{b = t\} \cap B_j) = 0$ for all t . Second, we define

$$b(s) = a(s) + d_{j+1} + \frac{d_j - d_{j+1}}{2}(1 - 2^{-k}), \quad s \in A_{j,k},$$

on each of the atoms $A_{j,k}$. In order to define b on the remainder B_j of the set $\{a = t_j\}$ we invoke Sierpiński's Theorem to obtain an increasing function $f_j : [0, \rho(B_j)] \rightarrow \mathcal{R}$ such that $\rho(f_j(y)) = y$ for all $y \in [0, \rho(B_j)]$ and define

$$b(s) = a(s) + d_{j+1} + \frac{d_j - d_{j+1}}{2} + \frac{d_j - d_{j+1}}{3\rho(B_j)} \cdot \inf\{\alpha : f_j(\alpha) \ni s\}, \quad s \in B_j.$$

Since b satisfies condition (ii) by construction, it is left to show that it satisfies (i) as well. To this end, let $x \in [0, \infty)$. Note that

$$\{b = x\} = \{a = x - d_j\}, \quad \text{whenever } \{b = x\} \cap C_j \neq \emptyset.$$

Since $t_j < x - d_j < t_{j-1}$, we know $\rho(\{b = x\}) = 0$. Furthermore, observe that if $\{b = x\} \cap D \neq \emptyset$, then $\rho(b = x) = \rho(a = x) = 0$. Second, note that if there exist indices $j \in \mathcal{N}$ and $k \in \mathcal{M}_j$ such that $\{b = x\} \cap A_{j,k} \neq \emptyset$, then $\{b = x\} = A_{j,k}$ by construction. Since $A_{j,k}$ is an atom, so is $\{b = x\}$. Third, assume there exists an index $j \in \mathcal{N}$ such that $\{b = x\} \cap B_j \neq \emptyset$. Note that by construction $\{b = x\} \subset B_j$, thus there exists some number x' such that

$$\{b = x\} = \{z \in B_j : \inf\{\alpha : f_j(\alpha) \ni z\} = x'\}.$$

Hence, for all $\varepsilon > 0$ we have

$$\{b = x\} \subset f_j(x' + \varepsilon) \setminus f_j(x' - \varepsilon),$$

and, by the properties of f_j , we conclude $\rho(b = x) = 0$. \square

Lemma 2.4. *Let $([0, \alpha), \mathcal{B}, \kappa)$ be a finite, signed measure space satisfying*

$$\kappa[0, t] \geq 0, \quad 0 \leq t \leq \alpha. \quad (2.3)$$

Let $f \in L_1(\kappa)$ be non-negative and decreasing. Then

$$\int_{[0, \alpha)} f(t) d\kappa(t) \geq 0.$$

Proof. First, observe that it is enough to show the assertion of the lemma for simple functions f of the form

$$f = \sum_{j=1}^m f_j \mathbb{1}_{B_j},$$

where $f_j \geq 0$ is decreasing and $\{B_j\}$ is a partition of $[0, \alpha)$ of measurable sets such that $\sup B_j = \inf B_{j+1}$, $1 \leq j \leq m-1$. Note that these conditions imply that B_j is a connected subset of $[0, \alpha)$. We define $g_j = \kappa(\bigcup_{i=1}^j B_i)$, $1 \leq j \leq m$ and $g_0 = 0$ and observe that (2.3) implies $g_j \geq 0$. This is true since $\bigcup_{i=1}^j B_i$ is either $[0, \sup B_j)$ or $[0, \sup B_j]$ and $\kappa[0, t] = \lim_n \kappa[0, t + 1/n] \geq 0$ for all $0 \leq t < \alpha$. Using partial summation we see

$$\begin{aligned} \int_{[0, \alpha)} f(t) d\kappa(t) &= \sum_{j=1}^m f_j \kappa(B_j) = \sum_{j=1}^m f_j (g_j - g_{j-1}) \\ &= f_m g_m - \sum_{j=1}^m (f_j - f_{j-1}) g_{j-1} \geq 0, \end{aligned}$$

since $f \geq 0$ is decreasing and $g_j \geq 0$ as noted before. \square

3. PROOF OF THE MAIN THEOREM

The purpose of this section is to prove Theorem 1.1. We start with some necessary definitions and lemmata.

3.1. Preparatory definitions and results. We define the measure space

$$(S, \Sigma, \sigma) := (\{1, \dots, n\} \times \Omega, \mathcal{P}(\{1, \dots, n\}) \otimes \mathcal{F}, \delta \otimes \mu),$$

where $(\Omega, \mathcal{F}, \mu)$ is the probability space from Theorem 1.1, δ is the counting measure on $\{1, \dots, n\}$ and $\mathcal{P}(\{1, \dots, n\})$ denotes the power set of $\{1, \dots, n\}$. Observe that $\sigma(S) = n$ and a measurable subset $A \subset S$ is an atom in S if and only if $A = \{i\} \times A'$ up to a σ -null set for some $1 \leq i \leq n$ and some atom A' in Ω .

Let $a : S \rightarrow \mathbb{R}$ be a measurable function with respect to Lebesgue measure on \mathbb{R} , which will be fixed throughout the entire section. Note that, without restriction, we assume that a is non-negative. We apply Proposition 2.3 to the function a and obtain a measurable function $b : S \rightarrow [0, \infty)$ with the following properties:

- (i) for all $x \in [0, \infty)$ we either have $\sigma(b = x) = 0$ or $\{b = x\}$ is an atom,
- (ii) for all $s, t \in S$, we have $a(s) > a(t)$ implies that $b(s) > b(t)$.

We define the set of all measurable functions on S that are ordered in the same way as b by

$$\begin{aligned} \mathcal{A}_b &:= \{d : S \rightarrow [0, \infty) \text{ measurable} \mid \\ &\quad b(x) \leq b(y) \implies d(x) \leq d(y) \text{ for all } x, y \in S\}. \end{aligned} \tag{3.1}$$

This means that, if b is constant on some set B , then any $d \in \mathcal{A}_b$ is constant on B as well. Note that in general, $d \in \mathcal{A}_b$ may be constant on some set B_0 , where b is not.

Moreover, we define the function $h : [0, \infty) \rightarrow \Sigma$ by

$$h(t) := \bigcup_{j=1}^{\infty} \{b \geq b^*(t - 1/j)\}.$$

Roughly speaking, $h(t)$ is the subset of S having approximately measure t , on which b takes its largest values. We single out those parameter values t such that $\sigma(h(t)) = t$ by setting

$$U := \{t \in [0, n] : \sigma(h(t)) = t\}.$$

Since U plays an important role in what follows, we first investigate some of its properties.

Lemma 3.1. *The set U has the following properties:*

- (i) $n \in U$,
- (ii) For all $t \in [0, n]$ we have that $t \in U^c$ if and only if there exists an open interval $V \ni t$ such that b^* is constant on V .
- (iii) U is closed.
- (iv) If $(c, d) \subset U^c$, then b^* is constant on (c, d) .

Proof. (i) By definition of h , and since b and b^* are equimeasurable, we have for all positive integers j :

$$\sigma(h(n)) \geq \sigma(b \geq b^*(n - 1/j)) = \lambda(b^* \geq b^*(n - 1/j)) \geq n - 1/j,$$

i.e., $\sigma(h(n)) \geq n$. On the other hand, $\sigma(h(n)) \leq \sigma(S) = n$.

(ii) Let $t \in U^c$. Thus, there exists an index j satisfying

$$\sigma(h(t)) \geq \sigma(b \geq b^*(t - 1/j)) > t.$$

This implies that $\lambda(b^* \geq b^*(t - 1/j)) > t$ and so there exist two points t_0, t_1 with $t_0 < t < t_1$ such that $b^*(t_0) = b^*(t_1)$. But since b^* is decreasing, b^* is constant on (t_0, t_1) . On the other hand, let b^* be constant on the interval $(t - 2\varepsilon, t + 2\varepsilon)$. This implies that $h(t) = h(t + \varepsilon)$ and so,

$$\sigma(h(t)) = \sigma(h(t + \varepsilon)) \geq t + \varepsilon > t,$$

i.e., $t \in U^c$.

(iii) This is an immediate consequence of (ii).

(iv) Finally, let $c < d$ with $(c, d) \subset U^c$ and $I \subset (c, d)$ be a compact interval. For every $t \in I$ we use (ii) to choose an open interval $V(t)$ containing t on which b^* is constant. By compactness, b^* is constant on I , and since I was arbitrary, (iv) is proved. \square

We now recall the assumptions of Theorem 1.1. The collection G is a subset of all mappings from $\{1, \dots, n\} \rightarrow \Omega$ and \mathbb{P} is a probability measure on G satisfying

- (i) $\mathbb{P}(g(i) \in A) = \mu(A)$,
- (ii) $\mathbb{P}(g(i_1) \in A_1, g(i_2) \in A_2) \leq C_G \mu(A_1) \mu(A_2)$

for all $i \in I$, all different indices $i_1, i_2 \in I$, and all measurable sets $A, A_1, A_2 \subset \Omega$. Next, for all $t \in [0, n]$, we define the random variable

$$X_t : G \rightarrow \{0, \dots, n\}, \quad g \mapsto |g \cap h(t)|,$$

where $|\cdot|$ denotes the cardinality and we interpret g as the graph of g , i.e. $g \cap h(t) = \{(i, g(i)) \in h(t) : i \in I\}$. Observe that the definition of h and property (iv) of the above lemma imply that for $t \in [0, n]$:

$$X_t = X_{u_0}, \quad \text{where } u_0 = \inf\{u \in U : u \geq t\}.$$

We will now study some properties of these random variables that are essential for the proof of Theorem 1.1.

Proposition 3.2. *The random variables (X_t) have the following properties:*

$$\begin{aligned}\mathbb{E}X_t &= \sigma(h(t)), & t \in [0, n], \\ \mathbb{E}X_t^2 &\leq t(1 + C_G t), & t \in U.\end{aligned}$$

Proof. Let $t \in [0, n]$. We have $X_t(g) = \sum_{i=1}^n Y_i^t(g)$, with $Y_i^t(g) := |\{(i, g(i))\} \cap h(t)| \in \{0, 1\}$ and, since $h(t)$ is a measurable subset of S , we can write

$$h(t) = \bigcup_{i=1}^n \{i\} \times A_i^t$$

with some measurable sets $A_1^t, \dots, A_n^t \subset \Omega$. Therefore, by assumption (i) in Theorem 1.1,

$$\mathbb{E}X_t = \sum_{i=1}^n \mathbb{E}Y_i^t = \sum_{i=1}^n \mathbb{P}(g(i) \in A_i^t) = \sum_{i=1}^n \mu(A_i^t) = \sigma(h(t)).$$

Now we assume $t \in U$ and estimate $\mathbb{E}X_t^2$ using assumption (ii) of Theorem 1.1:

$$\begin{aligned}\mathbb{E}X_t^2 &= \sum_{i=1}^n \mathbb{E}Y_i^t + \sum_{i \neq j} \mathbb{E}Y_i^t \cdot Y_j^t \\ &\leq t + \sum_{i \neq j} \mathbb{P}(g(i) \in A_i^t, g(j) \in A_j^t) \\ &\leq t + C_G \left(\sum_{i=1}^n \mu(A_i^t) \right)^2 = t(1 + C_G t),\end{aligned}$$

where we used that, by definition, $\sigma(h(t)) = t$ for any $t \in U$. \square

Proposition 3.3. *For all $t \in [0, n]$,*

$$\mathbb{P}(X_t \geq t/2) \geq \frac{t}{4 + 4C_G t}.$$

Proof. First, we assume that $t \in U$. Then, Paley-Zygmund's inequality (Theorem 2.1) in combination with Proposition 3.2 and the choice $\theta = 1/2$ imply the desired inequality. If $t \in U^c$, define $u_0 := \inf\{u \in U : u \geq t\}$. Hence, property (iv) of Lemma 3.1 implies $X_t = X_{u_0}$. Therefore,

$$\mathbb{P}(X_t \geq t/2) = \mathbb{P}(X_{u_0} \geq t/2) \geq \mathbb{P}(X_{u_0} \geq u_0/2) \geq \frac{u_0}{4 + 4C_G u_0}.$$

Since $u_0 \geq t$ and the function $s \mapsto s/(4 + 4C_G s)$ is increasing, the result follows. \square

Corollary 3.4. *For $t \in [0, n]$, we have*

$$\mathbb{P}(X_t \geq 1) \geq \min \left\{ \frac{t}{8}, \frac{1}{8C_G} \right\}.$$

Proof. If $t \leq 1/C_G$, we obtain from Proposition 3.3, the fact that X_t takes only integer values, and because $C_G \geq 1$

$$\mathbb{P}(X_t \geq 1) = \mathbb{P}(X_t \geq t/2) \geq \frac{t}{4 + 4C_G t} \geq \frac{t}{8}. \quad (3.2)$$

If $t > 1/C_G$, we get from Proposition 3.3 and since X_t takes only integer values

$$\mathbb{P}(X_t \geq 1) \geq \mathbb{P}(X_{1/C_G} \geq 1) = \mathbb{P}\left(X_{1/C_G} \geq \frac{1}{2C_G}\right) \geq \frac{1}{8C_G}. \quad (3.3)$$

Combining (3.2) and (3.3) concludes the proof. \square

As a matter of fact, we will use this corollary in the form

$$\mathbb{P}(X_t \geq 1) \geq \min \left\{ \frac{t}{8}, \frac{1}{8C_G} \right\} \mathbb{P}(X_\ell \geq 1), \quad 1 \leq \ell \leq n.$$

Corollary 3.5. *Let $k \in \mathbb{N}$ with $1 \leq k \leq n$. Then, for $t \in [2k, n]$,*

$$\mathbb{P}(X_t \geq k) \geq \frac{1}{2 + 4C_G}.$$

Proof. This is a direct consequence of Proposition 3.3. \square

Corollary 3.6. *For all $k \in \mathbb{N}$ in the range $1 \leq k \leq \ell/2$,*

$$\mathbb{E} \mathbf{k}\text{-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)} \geq \frac{1}{2 + 4C_G}.$$

Proof. Let k be an integer in the range $1 \leq k \leq \ell/2$. Using Proposition 3.3 with $t = 2k$, we obtain

$$\begin{aligned} \mathbb{E} \mathbf{k}\text{-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)} &\geq \mathbb{E}(\mathbf{k}\text{-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)}; X_{2k} \geq k) \\ &= \mathbb{P}(X_{2k} \geq k) \geq \frac{2k}{4 + 8C_G k} \geq \frac{1}{2 + 4C_G}. \end{aligned} \quad \square$$

3.2. Reduction to Boolean functions. In this section, we estimate the expression $\int_G \sum_{k=1}^\ell \mathbf{k}\text{-max}_{1 \leq i \leq n} |a(i, g(i))| d\mathbb{P}(g)$ occurring in Theorem 1.1 for general matrices a by the same expression with a replaced by some averaged matrix \tilde{a} . In order to begin our investigation, we first have to give a few definitions. For a measurable function $f \in \mathcal{A}_b$ we set

$$\tilde{f} := \frac{1}{\ell} \int_0^\ell f^*(s) ds \cdot \mathbb{1}_{h(\ell)} \quad \text{and} \quad f_t := \mathbb{1}_{h(t)}, \quad t \in [0, n]. \quad (3.4)$$

Observe that both $\tilde{f} \in \mathcal{A}_b$ and $f_t \in \mathcal{A}_b$. Moreover, we write

$$S_k(f)(g) := \mathbf{k}\text{-max}_{1 \leq i \leq n} f(i, g(i)) \quad \text{and} \quad S(f)(g) := \sum_{k=1}^\ell S_k(f)(g)$$

for all $f \in \mathcal{A}_b$ and $g \in G$. Then, for any $k \in \mathbb{N}$ with $1 \leq k \leq \ell$,

$$\mathbb{E} S_k(a_t) = \mathbb{P}(S_k(a_t) = 1) = \mathbb{P}(X_t \geq k), \quad t \in [0, n], \quad (3.5)$$

and, using also the equation in Proposition 3.2,

$$\mathbb{E} S_k(\tilde{a}_t) = \frac{\min\{\mathbb{E} X_t, \ell\}}{\ell} \mathbb{P}(X_\ell \geq k), \quad t \in [0, n], \quad (3.6)$$

where $\tilde{a}_t = (a_t)^\sim$, i.e. we first apply the operation \cdot_t and then the operation \sim .

We first establish our result for the special functions a_t in Proposition 3.8, which will then allow us to prove (cf. Subsection 3.3) the same result for general functions a in Theorem 3.9.

Proposition 3.7. *For all $t \in [0, n]$ we have*

$$\mathbb{E} S(a_t) \leq (6 + 12C_G) \cdot \mathbb{E} S(\tilde{a}_t).$$

Proof. First, assume $\ell = 1$. Then, equation (3.6) and Corollary 3.4 yield

$$\begin{aligned} \mathbb{E}S(\tilde{a}_t) &\geq \min\{\mathbb{E}X_t, 1\}\mathbb{P}(X_1 \geq 1) \\ &\geq \frac{1}{8C_G} \min\{\mathbb{E}X_t, 1\} \geq \frac{1}{8C_G} \mathbb{P}(X_t \geq 1) = \frac{1}{8C_G} \mathbb{E}S(a_t), \end{aligned}$$

where we used (3.5) in the latter equality. Second, assume that $\ell \geq 2$. Due to equation (3.6) we have

$$\mathbb{E}S(\tilde{a}_t) = \frac{\min\{\mathbb{E}X_t, \ell\}}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \geq \frac{\min\{\mathbb{E}X_t, \ell\}}{\ell} \sum_{k=1}^{\ell/2} \mathbb{P}(X_{\ell} \geq k).$$

Then, Corollary 3.5 and (3.5) give us

$$(6 + 12C_G) \cdot \mathbb{E}S(\tilde{a}_t) \geq \min\{\mathbb{E}X_t, \ell\} \geq \sum_{k=1}^{\ell} \mathbb{P}(X_t \geq k) = \mathbb{E}S(a_t),$$

where we used that

$$\mathbb{E}X_t = \sum_{k=1}^n \mathbb{P}(X_t \geq k). \quad \square$$

Proposition 3.8. *For all $t \in U$, we have*

$$\mathbb{E}S(\tilde{a}_t) \leq (8 + 16C_G) \cdot \mathbb{E}S(a_t).$$

Proof. Combining (3.5) and (3.6), we see that it is enough to prove the inequality

$$\frac{t}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \leq (8 + 16C_G) \cdot \sum_{k=1}^{\ell} \mathbb{P}(X_t \geq k) \quad (3.7)$$

for $t \in U \cap [0, \ell]$.

First, we assume that $t \leq 2$. Then, Corollary 3.4 implies

$$\frac{t}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \leq t \mathbb{P}(X_{\ell} \geq 1) \leq t \max\left\{\frac{8}{t}, 8C_G\right\} \mathbb{P}(X_t \geq 1).$$

Since $t \leq 2$, we further get

$$\frac{t}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \leq 16C_G \mathbb{P}(X_t \geq 1) \leq 16C_G \sum_{k=1}^{\ell} \mathbb{P}(X_t \geq k), \quad (3.8)$$

which implies (3.7) for $t \leq 2$.

Second, we assume $2m \leq t \leq 2(m+1)$ for some $1 \leq m \leq \ell/2$. In that case, Corollary 3.5 yields

$$\frac{t}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \leq \frac{t}{m} \sum_{k=1}^m \mathbb{P}(X_{\ell} \geq k) \leq \frac{t}{m} \sum_{k=1}^m (2 + 4C_G) \mathbb{P}(X_t \geq k).$$

Using the inequality $t \leq 2(m+1)$, we conclude

$$\frac{t}{\ell} \sum_{k=1}^{\ell} \mathbb{P}(X_{\ell} \geq k) \leq (8 + 16C_G) \cdot \sum_{k=1}^m \mathbb{P}(X_t \geq k) \leq (8 + 16C_G) \cdot \sum_{k=1}^{\ell} \mathbb{P}(X_t \geq k),$$

which is (3.7) for $t \geq 2$. Combining the latter with (3.8), the proof of the proposition is completed. \square

Theorem 3.9. *Let $a : S \rightarrow [0, \infty)$ be an arbitrary measurable function on S . Then,*

$$\frac{1}{6 + 12C_G} \cdot \mathbb{E}S(a) \leq \mathbb{E}S(\tilde{a}) \leq (8 + 16C_G) \cdot \mathbb{E}S(a). \quad (3.9)$$

Proof. Defining the \mathcal{F}^n -measurable function $u_k : G \rightarrow [0, n]$ by

$$u_k(g) := \inf\{t : X_t(g) \geq k\} = \inf\{t : |h(t) \cap g| \geq k\},$$

we first show that $d(f(u_k(g)))$ is well defined for all $d \in \mathcal{A}_b$, where $f(t) := b^{-1}(b^*(t))$, and that this expression actually satisfies

$$S_k(d)(g) = d(f(u_k(g))), \quad d \in \mathcal{A}_b.$$

For the definition of \mathcal{A}_b see (3.1). Observe that $u_k(g)$ is the unique number such that

$$|h(u_k(g) - \varepsilon) \cap g| < k \quad \text{and} \quad |h(u_k(g) + \varepsilon) \cap g| \geq k$$

for all $\varepsilon > 0$ and, additionally,

$$f(u_k(g)) = \bigcap_{\varepsilon > 0} h(u_k(g) + \varepsilon) \setminus h(u_k(g) - \varepsilon).$$

Hence, there exists an element $(i_0, g(i_0)) \in f(u_k(g))$ satisfying

$$b(i_0, g(i_0)) = S_k(b)(g). \quad (3.10)$$

Furthermore, observe that the definition of f implies

$$b(y) = b^*(u_k(g)) \quad \text{for all } y \in f(u_k(g)). \quad (3.11)$$

A consequence of the definition of \mathcal{A}_b is that (3.10) and (3.11) imply

$$d(i_0, g(i_0)) = S_k(d)(g) = d(y) \quad \text{for all } y \in f(u_k(g)).$$

Therefore, (3.11) together with the definition of \mathcal{A}_b gives us that $d \circ f : u_k(G) \rightarrow [0, \infty)$ is a well defined and decreasing function, thus measurable. Observe that by changing variables we obtain

$$\mathbb{E}S_k(d) = \int_G d(f(u_k(g))) \, d\mathbb{P}(g) = \int_{u_k(G)} d(f(z)) \, d\mathbb{P}_{u_k}(z).$$

We will now show that

$$\int_{u_k(G)} d(f(z)) \, d\mathbb{P}_{u_k}(z) = \int_{u_k(G)} d^*(z) \, d\mathbb{P}_{u_k}(z). \quad (3.12)$$

Without loss of generality, we may assume that d is a simple function of the form

$$d = \sum_{j=1}^m d_j \mathbb{1}_{D_j},$$

with (d_j) decreasing and $\{D_j\}$ a disjoint collection of measurable sets. Observe that in this case

$$d^* = \sum_{j=1}^m d_j \mathbb{1}_{[\sum_{i=1}^{j-1} \sigma(D_i), \sum_{i=1}^j \sigma(D_i))} \quad (3.13)$$

and (3.12) becomes

$$\sum_{j=1}^m d_j \int_{u_k(G)} \mathbb{1}_{D_j}(f(z)) \, d\mathbb{P}_{u_k}(z) = \sum_{j=1}^m d_j \mathbb{P}_{u_k} \left(\left[\sum_{i=1}^{j-1} \sigma(D_i), \sum_{i=1}^j \sigma(D_i) \right) \right).$$

Thus, it is sufficient to prove

$$u_k(G) \cap \{z : \{b = b^*(z)\} \subset D_j\} = u_k(G) \cap \left[\sum_{i=1}^{j-1} \sigma(D_i), \sum_{i=1}^j \sigma(D_i) \right). \quad (3.14)$$

On the one hand, let $z \in u_k(G) \cap \left[\sum_{i=1}^{j-1} \sigma(D_i), \sum_{i=1}^j \sigma(D_i) \right)$, hence, $d^*(z) = d_j$. Observe that since $d \in \mathcal{A}_b$ we have

$$\{y : b(y) = b^*(z)\} \subset \{y : d(y) = d^*(z)\} = D_j.$$

On the other hand, let $z \in u_k(G)$ be such that $\{y : b(y) = b^*(z)\} \subset D_j$. Note that there exists a unique index j_0 such that

$$\{y : b(y) = b^*(z)\} \subset \{y : d(y) = d^*(z)\} = D_{j_0}.$$

Observe that (3.10) implies that $u_k(G) \subset \{z : f(z) \neq \emptyset\}$, hence $\{y : b(y) = b^*(z)\} \neq \emptyset$. Thus, the disjointness of the $\{D_j\}$ implies $j = j_0$. Therefore, we obtain from (3.13) that $z \in \left[\sum_{i=1}^{j-1} \sigma(D_i), \sum_{i=1}^j \sigma(D_i) \right)$. This proves (3.14) and consequently (3.12). So far we proved that

$$\mathbb{E}S_k(d) = \int_{u_k(G)} d^*(z) d\mathbb{P}_{u_k}(z), \quad d \in \mathcal{A}_b.$$

Therefore, setting $\nu = \sum_{k=1}^\ell \mathbb{P}_{u_k}$ we obtain

$$\mathbb{E}S(d) = \int_{[0,n]} d^*(z) d\nu(z), \quad d \in \mathcal{A}_b.$$

Recalling (3.4) we observe that

$$\mathbb{E}S(\tilde{a}) = \frac{1}{\ell} \int_{[0,\ell]} a^*(s) ds \cdot \nu[0, \sigma(h(\ell))].$$

Having now introduced the necessary tools for the proof of (3.9), we first proceed by proving the upper estimate. Observe that with $C := 8 + 16C_G$, we can write

$$C\mathbb{E}S(a) - \mathbb{E}S(\tilde{a}) = \int_{[0,n]} a^*(x) d\tau(x) \quad (3.15)$$

with the signed measure $d\tau(x) = C d\nu(x) - \frac{\nu[0, \sigma(h(\ell))]}{\ell} d\eta(x)$, where η is the Lebesgue measure on $[0, \ell]$. We have shown in Proposition 3.8 that

$$C\nu[0, t] = C\mathbb{E}S(a_t) \geq \mathbb{E}S(\tilde{a}_t) = \frac{\min\{t, \ell\}}{\ell} \nu[0, \sigma(h(\ell))], \quad t \in U,$$

i.e., $\tau[0, t] \geq 0$ for all $t \in U$. We will now show that $\tau[0, t] \geq 0$ for all $t \in [0, n]$. To this end, let $t \in U^c$. Define $u_0 = \inf\{u \geq t : u \in U\}$ and note that $u_0 \in U$, since U is closed by Lemma 3.1. Observe that by (ii) of Lemma 3.1 $(t - \varepsilon, u_0) \subset U^c$ for some $\varepsilon > 0$. Hence, (iv) of Lemma 3.1 implies that b^* is constant on $(t - \varepsilon, u_0)$, which means by definition of h that $h(t) = h(u_0)$. As a consequence we obtain

$$C\nu[0, t] = C\nu[0, u_0] \geq \frac{\min\{u_0, \ell\}}{\ell} \nu[0, \sigma(h(\ell))] \geq \frac{\min\{t, \ell\}}{\ell} \nu[0, \sigma(h(\ell))],$$

i.e. $\tau[0, t] \geq 0$ for all $t \in [0, n]$. Applying Lemma 2.4 to the right hand side of (3.15) we obtain

$$C\mathbb{E}S(a) - \mathbb{E}S(\tilde{a}) \geq 0,$$

which concludes the proof of the upper estimate.

The proof of the lower estimate in (3.9) follows along the same lines by just employing Proposition 3.7 instead of Proposition 3.8 and using an appropriate signed measure different than τ . \square

3.3. Conclusion. As we have seen, we can reduce the case of general a to multiples of functions only taking values zero and one. We now use this fact to prove Theorem 1.1.

Proof of Theorem 1.1. First we prove the lower estimate. Observe that Theorem 3.9 and the definition of \tilde{a} yield

$$\begin{aligned} C_1 \cdot \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} a(i, g(i)) &\geq \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} \tilde{a}(i, g(i)) \\ &= \frac{1}{\ell} \int_0^{\ell} a^*(t) dt \cdot \left(\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)} \right), \end{aligned} \quad (3.16)$$

where $C_1 = 8(1 + 2C_G)$. If $\ell = 1$, then Corollary 3.4 implies

$$\begin{aligned} C_1 \mathbb{E} \max_{1 \leq i \leq n} a(i, g(i)) &\geq \int_0^1 a^*(t) dt \cdot \mathbb{E} \max_{1 \leq i \leq n} \mathbb{1}_{h(1)}(i, g(i)) \\ &\geq \int_0^1 a^*(t) dt \cdot \mathbb{P}(X_1 \geq 1) \\ &\geq \frac{1}{8C_G} \int_0^1 a^*(t) dt. \end{aligned}$$

For $\ell \geq 2$ we use Corollary 3.6 and see

$$\begin{aligned} C_1 \cdot \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} a(i, g(i)) &\geq \frac{1}{\ell} \int_0^{\ell} a^*(t) dt \cdot \left(\mathbb{E} \sum_{k=1}^{\ell/2} \text{k-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)}(i, g(i)) \right) \\ &\geq \frac{1}{6(1 + 2C_G)} \int_0^{\ell} a^*(t) dt, \end{aligned}$$

which proves the lower estimate.

Now, we proceed with the upper estimate. For all $\ell \geq 1$ we have by Theorem 3.9

$$\begin{aligned} c_1 \cdot \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} a(i, g(i)) &\leq \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} \tilde{a}(i, g(i)) \\ &= \frac{1}{\ell} \int_0^{\ell} a^*(t) dt \cdot \mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} \mathbb{1}_{h(\ell)}(i, g(i)) \\ &\leq \int_0^{\ell} a^*(t) dt, \end{aligned}$$

where $c_1 = 1/(6 + 12C_G)$. This concludes the proof of the theorem. \square

4. AN APPLICATION TO ORLICZ SPACES

We will present an application of our main result (cf. Theorem 1.1) dealing with averages of order statistics on random sequences. The expressions for the bounds on the expectations that we obtain for (1.1) are in terms of Orlicz norms and rather simple (cf. Theorem 1.4). Note that by our “direct” approach, we recover Corollaries 2 and 3 from [7].

We will estimate the following expression:

$$\mathbb{E} \sum_{k=1}^{\ell} \max_{1 \leq i \leq n} |x_i X_i|,$$

where X_1, \dots, X_n are independent copies of a random variable $X : (\Omega, \mu) \rightarrow \mathbb{R}$ with $\mathbb{E}|X| < \infty$. Those expressions were already studied in [7, 8]. There, the argument in the proof is built upon an estimate involving only the largest order statistic and combinatorial results that were obtained in [13, 14]. However, with Theorem 1.1, problems of this form can be approached directly.

Recall that M^* in Theorem 1.4 is defined by

$$M^*\left(\int_0^\beta X^*(y) dy\right) = \frac{\beta}{\ell}, \quad 0 \leq \beta \leq 1.$$

The following Lemma is a continuous version of Lemma 2.1 in [13] and our proof follows along the same lines.

Lemma 4.1. *Let*

$$B = \text{conv} \left\{ \left(\varepsilon_i \int_0^{\alpha_i} X^*(y) dy \right)_{i=1}^n : \varepsilon_i = \pm 1, \sum_{i=1}^n \alpha_i = \ell \right\}. \quad (4.1)$$

Then we have

$$B \subset B_{M^*} \subset 3B.$$

Proof. First, we show the left inclusion. Let $z \in B$. Then

$$\sum_{i=1}^n M^*(|z_i|) = \sum_{i=1}^n M^*\left(\int_0^{\alpha_i} X^*(y) dy\right) = \sum_{i=1}^n \alpha_i / \ell = 1.$$

To show the other inclusion, let $z_1 \geq \dots \geq z_n > 0$ and

$$\sum_{i=1}^n M^*(z_i) = 1.$$

We write $z = z' + z'' = (z_1, \dots, z_r, 0, \dots, 0) + (0, \dots, 0, z_{r+1}, \dots, z_n)$, where r is chosen such that $M^*(z_i) > 1/n$ for all $1 \leq i \leq r$, and $M^*(z_i) \leq 1/n$ for all $r \geq i+1$.

We have

$$M^*\left(\int_0^{\ell/n} X^*(y) dy\right) = \frac{1}{n}.$$

Therefore, $z'' \leq (\int_0^{\ell/n} X^*(y) dy, \dots, \int_0^{\ell/n} X^*(y) dy) =: w \in \mathbb{R}^n$. Since $w \in B$, we also have $z'' \in B$.

It is now left to show that $z' \in B$. There exist indices $k_i \geq 1$ for $1 \leq i \leq r$ such that

$$\frac{k_i}{n} \leq M^*(z_i) \leq \frac{k_i + 1}{n}. \quad (4.2)$$

Since

$$\sum_{i=1}^r \frac{k_i}{n} = \sum_{i=1}^r M^* \left(\int_0^{\ell k_i/n} X^*(y) dy \right) \leq \sum_{i=1}^r M^*(z_i) \leq 1,$$

and $\sum_{i=1}^r \ell k_i/n \leq \ell$, we immediately obtain

$$\left(\int_0^{\ell k_1/n} X^*(y) dy, \dots, \int_0^{\ell k_r/n} X^*(y) dy, 0, \dots, 0 \right) \in B.$$

Using (4.2), we see that

$$\begin{aligned} 2z &= \left(2 \int_0^{\ell k_1/n} X^*(y) dy, \dots, 2 \int_0^{\ell k_r/n} X^*(y) dy, 0, \dots, 0 \right) \\ &\geq \left(\int_0^{2\ell k_1/n} X^*(y) dy, \dots, \int_0^{2\ell k_r/n} X^*(y) dy, 0, \dots, 0 \right) \geq z'. \end{aligned}$$

Thus $z' \in 2B$. We conclude that $z \in 3B$. \square

Proof of Theorem 1.4. We will assume that the independence of X_1, \dots, X_n is realized through n factors with $G = \Omega^n$ and $X_i(g) = X(g(i))$ for $g \in G$ and a canonical random variable X with the same distribution as X_1, \dots, X_n . This first means that

$$\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X_i| = \mathbb{E}_G \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X(g(i))|.$$

Defining $a : \{1, \dots, n\} \times \Omega \rightarrow \mathbb{R}$ by

$$a(i, \omega) := x_i X(\omega),$$

and setting $\mathbb{P} = \bigotimes_{i=1}^n \mu$, we obtain by Theorem 1.1

$$\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X_i| = \int_G \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |a(i, g(i))| d\mathbb{P}(g) \simeq \int_0^{\ell} a^*(t) dt.$$

Observe that we also have

$$\int_0^{\ell} a^*(t) dt = \sup_{\sum \alpha_i = \ell} \sum_{i=1}^n x_i \int_0^{\alpha_i} X^*(t) dt,$$

by approximation of X with simple functions. Therefore,

$$\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X_i| \simeq \sup_{\sum \alpha_i = \ell} \sum_{i=1}^n x_i \int_0^{\alpha_i} X^*(t) dt.$$

With B as in (4.1), using Lemma 4.1 and (2.1), we further obtain

$$\mathbb{E} \sum_{k=1}^{\ell} \text{k-max}_{1 \leq i \leq n} |x_i X_i| \simeq \sup_{y \in \text{ext} B} \sum_{i=1}^n x_i y_i \simeq \sup_{y \in B_{M^*}} \sum_{i=1}^n x_i y_i \simeq \|x\|_M.$$

This concludes the proof. \square

5. AN APPLICATION TO THE LOCAL THEORY OF BANACH SPACES

In this last section we present an example of a set of maps with a minimal number of elements satisfying conditions (i) and (ii) in Corollary 1.2. This is then used to embed certain Orlicz sequence spaces ℓ_M^n into $\ell_1^{cn^3}$ using the “standard” embedding, which usually provides an embedding into $\ell_1^{n!2^n}$.

Recall that given two normed spaces X, Y and some constant $C \geq 1$, we say that X C -embeds into Y and write $X \xrightarrow{C} Y$ if there exists a one to one linear operator $\Psi : X \rightarrow \Psi(X) \subseteq Y$ such that $\|\Psi\| \cdot \|\Psi^{-1}\| \leq C$. Given two isomorphic Banach spaces X and Y the Banach-Mazur distance of X and Y is defined as

$$d_{\text{BM}}(X, Y) = \inf \{ \|T\| \|T^{-1}\| : T \in L(X, Y) \text{ isomorphism} \}.$$

Before we continue, let us give some historical remarks. The problem, given an n -dimensional subspace X of $L_1([0, 1], dx)$ and $\varepsilon > 0$, what is the smallest $N = N(X, \varepsilon)$ such that there is a subspace Y of ℓ_1^N with $d_{\text{BM}}(X, Y) \leq 1 + \varepsilon$, is extensively studied in the literature. A first breakthrough was made by G. Schechtman in [28], proving that

$$N \leq \frac{C}{\varepsilon^2} \log(\varepsilon^{-1}) \cdot n^2,$$

$C > 0$ being an absolute constant.

Based on his work, in [2] the bound in the dimension was improved to

$$N \leq \frac{C}{\varepsilon^2} \log(n\varepsilon^{-1})(\log n)^2 \cdot n$$

Later, M. Talagrand in [31] proved that for any $0 < \varepsilon \leq \varepsilon_0$

$$N \leq CK(X)^2 \varepsilon^{-2} \cdot n, \tag{5.1}$$

where $K(X)$ denotes the K -convexity constant of X (cf. [20]). Recall that G. Pisier proved in [22] that if $X \subseteq L_1([0, 1], dx)$ and $\dim(X) = n$, then $K(X) \leq C\sqrt{\log n}$. Thus (5.1) improves on previous results and gives

$$N \leq C\varepsilon^{-2} \log(n)n. \tag{5.2}$$

For further information we also refer to the work of W. B. Johnson and G. Schechtman [12], A. Naor and A. Zvavitch [21], J. Bernués and M. López-Valdes, O. Friedland and O. Guédon, as well as the references therein.

Now, recall that in [3], using the theorem of de Finetti, J. Bretagnolle and D. Dacunha-Castelle proved that an Orlicz space ℓ_M is isomorphic to a subspace of L_1 if and only if M is equivalent to a 2-concave Orlicz function. The corresponding finite-dimensional result was proved by S. Kwapien and C. Schütt in [13, 30]. In combination with the previous results, this shows that ℓ_M^n C -embeds into $\ell_1^{c \log(n)n}$.

The proofs of the results mentioned above are very involved and quite technical. The embedding we present here is specific and rather simple, with an offset in the dimension N .

5.1. A minimal set of maps. Let n be a power of a prime number and let \mathbb{F}_n denote the field with n elements. We define $I_0 = \Omega_0 = \mathbb{F}_n$ and denote by μ_0 the probability measure on Ω_0 defined by $\mu_0(\{i\}) = \frac{1}{n}$, for all $i \in \mathbb{F}_n$. If we set $G_0 = \{g_{\ell m} : \ell, m \in \mathbb{F}_n\}$, where $g_{\ell m}(i) = \ell i + m$ and multiplication and addition is performed in \mathbb{F}_n , then the probability measure \mathbb{P}_0 on G_0 given by $\mathbb{P}_0(\{g\}) = \frac{1}{n^2}$, for all $g \in G_0$ satisfies conditions (i) and (ii) of Theorem 1.1 with $C_{G_0} = 1$, that is

- (i) $\mathbb{P}_0(g(i) = j) = \frac{1}{n}$, for all $(i, j) \in I_0 \times \Omega_0$,
- (ii) $\mathbb{P}_0(g(i_1) = j_1, g(i_2) = j_2) \leq \frac{1}{n^2}$, for all $(i_1, j_1) \neq (i_2, j_2)$ in $I_0 \times \Omega_0$.

We want to point out that the above G_0 consists of n^2 elements.

We first prove condition (i) and let $(i, j) \in I_0 \times \Omega_0$. Then, for an arbitrary $\ell \in \mathbb{F}_n$, there exists exactly one $m \in \mathbb{F}_n$, which is given by $m = j - \ell i$ such that $g_{\ell m}(i) = \ell i + m = j$. Therefore, condition (i) is satisfied. For condition (ii) we note that for all different tuples $(i_1, j_1), (i_2, j_2) \in I_0 \times \Omega_0$, in order to have $g_{\ell m}(i_1) = \ell i_1 + m = j_1$ and $g_{\ell m}(i_2) = \ell i_2 + m = j_2$ for some $\ell, m \in \mathbb{F}_n$, it is necessary that $i_1 \neq i_2$ and in this case ℓ is given (uniquely) by $\ell = (j_1 - j_2)(i_1 - i_2)^{-1}$ and $m = j_1 - \ell i_1 = j_2 - \ell i_2$. Therefore, the event $\{g \in G_0 : g(i_1) = j_1, g(i_2) = j_2\}$ consists of at most one element and the definition of \mathbb{P}_0 implies condition (ii).

In general, we have the following result. Let $n \in \mathbb{N}$, define $I_1 = \Omega_1 = \{1, \dots, n\}$ and set $\mu_1(\{i\}) = \frac{1}{n}$, for all $1 \leq i \leq n$. If G_1 and \mathbb{P}_1 satisfy conditions (i) and (ii) of Theorem 1.1 for some constant $C_{G_1} \geq 1$, then G_1 consists of at least $\frac{n^2}{C_{G_1}}$ elements.

Indeed, assume that conditions (i) and (ii) are satisfied with some constant $C_{G_1} \geq 1$ and assume that G_1 consists of less than n^2/C_{G_1} elements. Then, there exists at least one element $g_1 \in G_1$ such that $\mathbb{P}(\{g_1\}) > C_{G_1}/n^2$. Since, for the choice $j_1 = g_1(i_1), j_2 = g_1(i_2)$ and arbitrary different $i_1, i_2 \in \mathbb{F}_n$, this g is an element of the event $\{g \in G_1 : g(i_1) = j_1, g(i_2) = j_2\}$. Therefore, by (ii), we get the contradiction

$$\frac{C_{G_1}}{n^2} < \mathbb{P}(\{g_1\}) \leq \mathbb{P}(\{g \in G_1 : g(i_1) = j_1, g(i_2) = j_2\}) \leq \frac{C_{G_1}}{n^2}$$

This shows that, up to a constant factor, the set of functions G_0 has the least number of elements satisfying conditions (i) and (ii) of Theorem 1.1.

5.2. Embedding ℓ_M^n into $\ell_1^{cn^3}$. As an application to Banach space theory, we will now apply Theorem 1.1 to $I_0, \Omega_0, \mu_0, G_0$ and \mathbb{P}_0 , defined as above, and prove Theorem 1.3. We start by explaining the rough idea before going through the details. Let M be a strictly convex, twice differentiable Orlicz function that is strictly 2-concave. We will show that the Orlicz sequence space ℓ_M^n C -embeds into $\ell_1^{cn^3}$, where c and C are absolute constants independent of n and M . This should be compared with the “standard” embedding of ℓ_M^n into $\ell_1^{n!2^n}$. Recall that the standard embedding (cf. [30]) is given by

$$\Psi_n : \ell_M^n \rightarrow \ell_1^{n!2^n}, \quad x \mapsto \frac{1}{n!2^n} \left(\sum_{i=1}^n \varepsilon_i a_{\pi(i)} x_i \right)_{\pi, \varepsilon},$$

where $a = a(M) \in \mathbb{R}^n$ is chosen in such a way that it generates the Orlicz norm, i.e.,

$$\frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \left(\sum_{i=1}^n |x_i a_{\pi(i)}|^2 \right)^{1/2} \simeq \|x\|_M.$$

Indeed, using Khintchine's inequality we then obtain that

$$\begin{aligned} \|\Psi_n(x)\|_1 &= \frac{1}{n!2^n} \sum_{\pi, \varepsilon} \left| \sum_{i=1}^n \varepsilon_i a_{\pi(i)} x_i \right| \\ &\simeq \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \left(\sum_{i=1}^n |x_i a_{\pi(i)}|^2 \right)^{1/2} \\ &\simeq \|x\|_M. \end{aligned}$$

So the standard embedding combines Khintchine's inequality with an average over the symmetric group \mathfrak{S}_n , which explains the dimension $n!2^n$ (see also [29, 25, 23, 24] for embeddings of other types of spaces into L_1). Instead of taking an average over the whole symmetric group, we rather use our minimal set of maps G_0 , which has cardinality n^2 only, thus obtaining an embedding into $\ell_1^{n^2 2^n}$. To further decrease the dimension, we will then make use of a result due to J. Bourgain, J. Lindenstrauss, and V.D. Milman that allows to use only cn sign vectors instead of 2^n .

Now let us be more precise. To find the sequence $(a_i)_{i=1}^n$ of scalars that generates the Orlicz norm, we use the following result due to C. Schütt (cf. [30, Theorem 2]): if M is a strictly convex, twice differentiable Orlicz function that is strictly 2-concave and satisfies $M^*(1) = 1$, then there exists a sequence a_1, \dots, a_n of scalars such that for all $x \in \mathbb{R}^n$,

$$\frac{1}{c} \|x\|_M \leq \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \left(\sum_{i=1}^n |x_i a_{\pi(i)}|^2 \right)^{1/2} \leq c \|x\|_M, \quad (5.3)$$

where c is a constant that does not depend on n and M . As a matter of fact, in [30, Theorem 2] an explicit formula for the choice of a is given.

J. Bourgain, J. Lindenstrauss, and V. D. Milman [1] proved the following: if v_1, \dots, v_n are unit vectors in a normed space $(\mathbb{R}^n, \|\cdot\|)$, then, for all $\delta > 0$, there exists a constant $C(\delta) > 0$ and $N = C(\delta)n$ sign vectors $\varepsilon^1, \dots, \varepsilon^N \in \{-1, 1\}^n$ such that for all $x \in \mathbb{R}^n$

$$(1 - \delta) \text{Ave}_{\pm} \left\| \sum_{i=1}^n \pm x_i v_i \right\| \leq \frac{1}{N} \sum_{j=1}^N \left\| \sum_{i=1}^n \varepsilon_i^j x_i v_i \right\| \leq (1 + \delta) \text{Ave}_{\pm} \left\| \sum_{i=1}^n \pm x_i v_i \right\|. \quad (5.4)$$

For our purpose, it is enough to know that in the setting $v_1 = v_2 = \dots = v_n = e_1$, where e_1 is the first standard unit vector of \mathbb{R}^n , $\|\cdot\| = \|\cdot\|_1$, and say $\delta = 1/4$, there exists a choice of N sign vectors that satisfy (5.4).

The last ingredient is a special case of a result we recently obtained in [16, Theorem 1.4]) and reads as follows: let $n \in \mathbb{N}$, $a \in \mathbb{R}^{n \times n}$, and $1 \leq p < \infty$. Let G be a collection of maps from $I = \{1, \dots, n\}$ to I and $C_G > 0$ be a constant only depending on G . Assume that for all $i, j \in I$ and all different pairs $(i_1, j_1), (i_2, j_2) \in I \times I$

- (i) $\mathbb{P}(\{g \in G : g(i) = j\}) = 1/n$,
- (ii) $\mathbb{P}(\{g \in G : g(i_1) = j_1, g(i_2) = j_2\}) \leq C_G/n^2$.

Then

$$\begin{aligned} C \left[\frac{1}{n} \sum_{k=1}^n s(k) + \left(\frac{1}{n} \sum_{k=n+1}^{n^2} s(k)^p \right)^{1/p} \right] &\leq \mathbb{E} \left(\sum_{i=1}^n |a_{ig(i)}|^p \right)^{1/p} \\ &\leq \frac{1}{n} \sum_{k=1}^n s(k) + \left(\frac{1}{n} \sum_{k=n+1}^{n^2} s(k)^p \right)^{1/p}, \end{aligned}$$

where $(s(k))_{k=1}^{n^2}$ is the decreasing rearrangement of $\{|a_{ij}| : i, j = 1, \dots, n\}$ and $C > 0$ is a constant only depending on C_G .

Note that the conditions in the theorem are satisfied for $G = G_0$, as was shown above, and for $G = \mathfrak{S}_n$ (cf. [16, Example 1.2]), although with different, but still absolute constants. This means that, especially for $p = 2$,

$$\frac{1}{n^2} \sum_{g \in G_0} \left(\sum_{i=1}^n |a_{ig(i)}|^2 \right)^{1/2} \simeq \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \left(\sum_{i=1}^n |a_{i\pi(i)}|^2 \right)^{1/2} \quad (5.5)$$

for all $a \in \mathbb{R}^{n \times n}$.

Let us now prove the embedding result.

Proof of Theorem 1.3. Let G_0 be our minimal set of maps. We define the isomorphism Ψ_n by

$$\Psi_n : \ell_M^n \rightarrow \ell_1^{Cn^3}, \quad x \mapsto \frac{1}{Cn^3} \left(\sum_{i=1}^n \varepsilon_i^j a_{g(i)} x_i \right)_{g \in G_0, j=1, \dots, Cn}.$$

Then a direct computation as shown in the standard embedding, now using equations (5.3), (5.4) (in the setting mentioned above), and (5.5), shows that

$$\|\Psi_n(x)\|_1 \simeq \|x\|_M.$$

This means that there exist absolute constants $C, C_1 > 0$ such that for all $n \in \mathbb{N}$, ℓ_M^n C_1 -embeds into $\ell_1^{Cn^3}$, where $C, C_1 > 0$ are independent of M . \square

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